COUPLED PAINLEVÉ III SYSTEMS WITH AFFINE WEYL GROUP SYMMETRY OF TYPES $B_5^{(1)}, D_5^{(1)}$ AND $D_6^{(2)}$

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Abstract. We find and study four kinds of five-parameter family of six-dimensional coupled Painlevé III systems with affine Weyl group symmetry of types $D_5^{(1)}, B_5^{(1)}$ and $D_6^{(2)}$. We show that each system is equivalent by an explicit birational and symplectic transformation, respectively. We also show that we characterize each system from the viewpoint of holomorphy.

1. Introduction

In [3, 4, 5, 6, 7], we presented some types of coupled Painlevé systems with various affine Weyl group symmetries. In this paper, we present a 5-parameter family of coupled Painlevé III systems with affine Weyl group symmetry of type $D_5^{(1)}$, which is explicitly given by

$$\frac{dx}{dt}=\frac{\partial H}{\partial y},\quad \frac{dy}{dt}=-\frac{\partial H}{\partial x},\quad \frac{dz}{dt}=\frac{\partial H}{\partial w},\quad \frac{dw}{dt}=-\frac{\partial H}{\partial z},\quad \frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q} \tag{1}$$

with the Hamiltonian

$$\begin{aligned} H=&H_1(x,y,t;\alpha_0,\alpha_1)+H_{III}^{D_7^{(1)}}(z,w,t;\alpha_0+\alpha_1+2\alpha_2)\\ &+H_4(q,p,t;\alpha_4,\alpha_5)+\frac{2(xz-wp)}{t}\\ =&\frac{x^2y^2+xy^2-(\alpha_0+\alpha_1)xy-\alpha_0y}{t}+\frac{z^2w^2+(\alpha_0+\alpha_1+2\alpha_2)zw+z+tw}{t}\\ &+\frac{q^2p^2-tq^2p-(1-\alpha_4-\alpha_5)qp-\alpha_4tq}{t}+\frac{2(xz-wp)}{t}. \end{aligned} \tag{2}$$

Here x,y,z,w,q and p denote unknown complex variables, and $\alpha_0,\alpha_1,\dots,\alpha_5$ are complex parameters satisfying the relation:

$$\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5=1. \tag{3}$$

The symbols H_1,H_2,H_3,H_4 and $H_{III}^{D_7^{(1)}}$ denote the polynomial Hamiltonians explicitly given as follows:

$$H_1=H_1(q,p,t;\alpha_0,\alpha_1)=\frac{q^2p^2+qp^2-(\alpha_0+\alpha_1)qp-\alpha_0p}{t}\quad(\alpha_0-\alpha_1+2\alpha_2=0), \tag{4}$$

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(5)
$$H_2=H_2(q,p,t;\alpha_2)=\frac{q^2p^2+(1-2\alpha_2)qp+tp}{t} \quad (\alpha_0-\alpha_1+2\alpha_2=0),$$
(6)
$$H_3=H_3(q,p,t;\alpha_2)=\frac{q^2p^2+2\alpha_2qp-q}{t} \quad (\alpha_0-\alpha_1+2\alpha_2=0),$$
(7)
$$H_4=H_4(q,p,t;\alpha_0,\alpha_1)=\frac{q^2p^2-tq^2p-(1-\alpha_0-\alpha_1)qp-\alpha_0tq}{t} \quad (\alpha_0-\alpha_1+2\alpha_2=0),$$
(8)
$$H_{III}^{D_7^{(1)}}=H_{III}^{D_7^{(1)}}(q,p,t;\alpha_1)=\frac{q^2p^2+\alpha_1qp+q+tp}{t} \quad (\alpha_0+\alpha_1=1).$$

We remark that for $y=q/\tau,\ t=\tau^2$ the Hamiltonian system with $H_{III}^{D_7^{(1)}}$ is the special case of the third Painlevé system (see [9]):

$$\frac{d^2y}{d\tau^2}=\frac{1}{y}\left(\frac{dy}{d\tau}\right)^2-\frac{1}{\tau}\frac{dy}{d\tau}+\frac{1}{\tau}(ay^2+b)+cy^3+\frac{d}{y} \tag{9}$$

with

$$a=-8, \quad b=4(1-\alpha_1), \quad c=0, \quad d=-4. \tag{10}$$

From the viewpoint of symmetry, the Hamiltonian system

$$\frac{dq}{dt}=\frac{\partial H_{III}^{D_7^{(1)}}}{\partial p}, \quad \frac{dp}{dt}=-\frac{\partial H_{III}^{D_7^{(1)}}}{\partial q} \tag{11}$$

has extended affine Weyl group symmetry of type $A_1^{(1)}$, whose generators $<s_0,s_1,\pi=\sigma\circ s_1>$ are explicitly given as follows (see [9]):

$$\begin{cases} s_0(q,p,t;\alpha_0,\alpha_1)=(q,p+\dfrac{\alpha_0}{q}-\dfrac{t}{q^2},-t;-\alpha_0,\alpha_1+2\alpha_0), \\ s_1(q,p,t;\alpha_0,\alpha_1)=(-q+\dfrac{\alpha_1}{p}+\dfrac{1}{p^2},-p,-t;\alpha_0+2\alpha_1,-\alpha_1), \\ \sigma(q,p,t;\alpha_0,\alpha_1)=(tp,-\dfrac{q}{t},-t;\alpha_1,\alpha_0). \end{cases} \tag{12}$$

PROPOSITION 1.1. *By the following birational and symplectic transformations tr_i $(i=1,2,3)$:*

$$\begin{cases} tr_1(q,p)=(t/p,(qp-\alpha_0)p/t), \\ tr_2(q,p)=(-tp,q/t), \\ tr_3(q,p)=(p/t,-tq), \end{cases} \tag{13}$$

the Hamiltonians H_1,H_2,H_3 and H_4 satisfy the following relations:

$$tr_1(H_1)=H_2, \quad tr_2\circ tr_1(H_1)=H_3, \quad tr_3(H_1)=H_4. \tag{14}$$

By Proposition 1.1, we see that each Hamiltonian H_i $(i = 1, 2, 3, 4)$ is equivalent by the transformations (13).

The Hamiltonian system with H_1 has the first integral.

PROPOSITION 1.2. *The system with the Hamiltonian* H_1

$$\frac{dq}{dt} = \frac{\partial H_1}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_1}{\partial q} \tag{15}$$

has the first integral I:

$$I = q^2p^2 + qp^2 - (\alpha_0 + \alpha_1)qp - \alpha_0 p. \tag{16}$$

We see that the relation between the Hamiltonian H_1 and the first integral I is explicitly given by

$$tH_1 = I. \tag{17}$$

The Bäcklund transformations of this system satisfy Noumi-Yamada's universal description for $D_5^{(1)}$ root system (see [1]). Since these universal Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry. The aim of this paper is to introduce the system of type $D_5^{(1)}$.

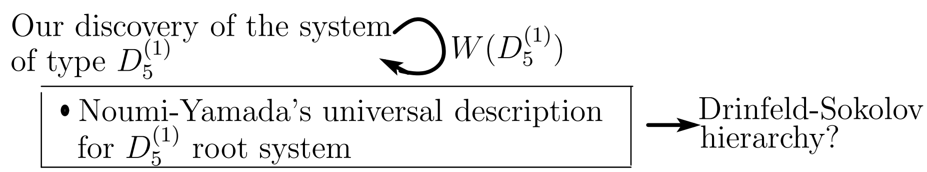

FIGURE 1.

We also show that this system coincides with some types of 5-parameter family of six-dimensional coupled Painlevé III systems with extended affine Weyl group symmetry of types $B_5^{(1)}$ and $D_6^{(2)}$. Moreover, we show the relationship between the system of type $D_5^{(1)}$ and the system of types $B_5^{(1)}$ and $D_6^{(2)}$ by an explicit birational and symplectic transformation, respectively.

This paper is organized as follows. In Section 2, we will introduce the system of type $D_5^{(1)}$ and its Bäcklund transformations. In Section 3, we will propose two types of a 5-parameter family of coupled Painlevé III systems in dimension six with extended affine Weyl group symmetry of type $B_5^{(1)}$. We also show that each of them is equivalent to the system (1) by a birational and symplectic transformation, respectively. In Section 4, we will propose a 5-parameter family of coupled Painlevé III systems in dimension six with extended affine Weyl group symmetry of type $D_6^{(2)}$ and its Bäcklund transformations. We also show that this system is equivalent to the system (1) by a birational and symplectic transformation.

2. The system of type $D_5^{(1)}$

In this section, we present a 5-parameter family of polynomial Hamiltonian systems that can be considered as six-dimensional coupled Painlevé III systems given by

$$
(18)\qquad
\begin{cases}
\dfrac{dx}{dt} = \dfrac{2x^2y+2xy-(\alpha_0+\alpha_1)x-\alpha_0}{t},\\
\dfrac{dy}{dt} = -\dfrac{2xy^2+y^2-(\alpha_0+\alpha_1)y+2z}{t},\\
\dfrac{dz}{dt} = \dfrac{2z^2w+(\alpha_0+\alpha_1+2\alpha_2)z+t-2p}{t},\\
\dfrac{dw}{dt} = -\dfrac{2zw^2+(\alpha_0+\alpha_1+2\alpha_2)w+1+2x}{t},\\
\dfrac{dq}{dt} = \dfrac{2q^2p-tq^2-(1-\alpha_4-\alpha_5)q-2w}{t},\\
\dfrac{dp}{dt} = \dfrac{-2qp^2+2tqp+(1-\alpha_4-\alpha_5)p+\alpha_4 t}{t}
\end{cases}
$$

with the Hamiltonian (2).

Theorem 2.1. *The system* (18) *admits extended affine Weyl group symmetry of type* $D_5^{(1)}$ *as the group of its Bäcklund transformations* (cf. [2, 8, 9]), *whose generators are explicitly given as follows*: *with the notation* $(*) := (x, y, z, w, q, p, t; \alpha_0, \alpha_1, \dots, \alpha_5)$,

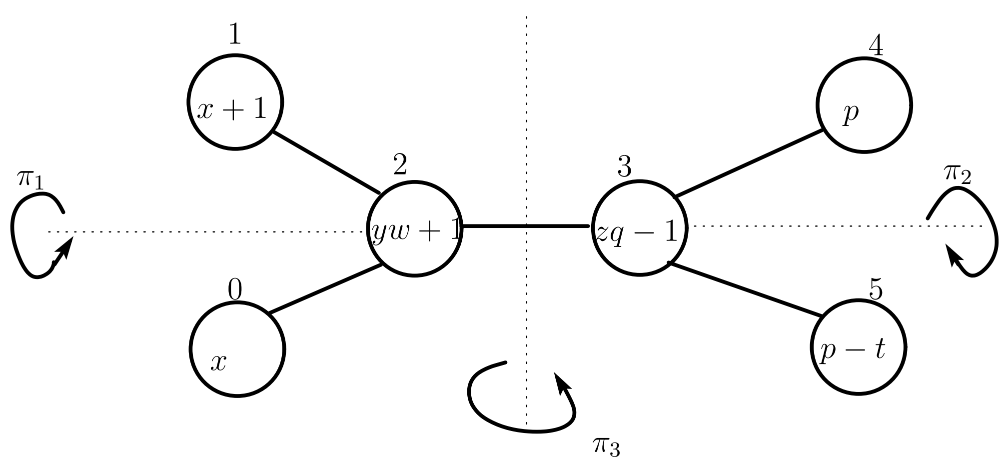

Figure 2. Dynkin diagram of type $D_5^{(1)}$

$$
s_0 : (*) \to \left(x, y-\frac{\alpha_0}{x}, z, w, q, p, t; -\alpha_0, \alpha_1, \alpha_2+\alpha_0, \alpha_3, \alpha_4, \alpha_5\right),
$$
$$
s_1 : (*) \to \left(x, y-\frac{\alpha_1}{x+1}, z, w, q, p, t; \alpha_0, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5\right),
$$

$$s_2:(*)\rightarrow\left(x+\frac{\alpha_2 w}{yw+1},y,z+\frac{\alpha_2 y}{yw+1},w,q,p,t;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4,\alpha_5\right),$$
$$s_3:(*)\rightarrow\left(x,y,z,w-\frac{\alpha_3 q}{zq-1},q,p-\frac{\alpha_3 z}{zq-1},t;\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3\right),$$
$$s_4:(*)\rightarrow\left(x,y,z,w,q+\frac{\alpha_4}{p},p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5\right),$$
$$s_5:(*)\rightarrow\left(x,y,z,w,q+\frac{\alpha_5}{p-t},p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_5,\alpha_4,-\alpha_5\right),$$
$$\pi_1:(*)\rightarrow(-x-1,-y,-z,-w,-q,-p,-t;\alpha_1,\alpha_0,\alpha_2,\alpha_3,\alpha_4,\alpha_5),$$
$$\pi_2:(*)\rightarrow(x,y,z,w,q,p-t,-t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_4),$$
$$\pi_3:(*)\rightarrow\left(\frac{(p-t)}{t},-tq,-tw,\frac{z}{t},\frac{y}{t},-t(x+1),-t;\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0\right).$$

THEOREM 2.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H\in\mathbb{C}(t)[x,y,z,w,q,p]$. *We assume that*

(A1) *deg*$(H)=4$ *with respect to* x,y,z,w,q,p.

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate* r_i $(i=0,1,4,5)$:

$$\begin{aligned}
&r_0:x_0=-(xy-\alpha_0)y,\ y_0=1/y,\ z_0=z,\ w_0=w,\ q_0=q,\ p_0=p,\\
&r_1:x_1=-((x+1)y-\alpha_1)y,\ y_1=1/y,\ z_1=z,\ w_1=w,\ q_1=q,\ p_1=p,\\
&r_4:x_4=x,\ y_4=y,\ z_4=z,\ w_4=w,\ q_4=1/q,\ p_4=-(pq+\alpha_4)q,\\
&r_5:x_5=x,\ y_5=y,\ z_5=z,\ w_5=w.\ q_5=1/q,\ p_5=-((p-t)q+\alpha_5)q.
\end{aligned}$$

(A3) *In addition to the assumption* (A2), *the Hamiltonian system in each coordinate system* $(x_i,y_i,z_i,w_i,q_i,p_i)$ $(i=0,4)$ *becomes again a polynomial Hamiltonian system in each coordinate* r_i $(i=2,3)$, *respectively*:

$$\begin{aligned}
&r_2:x_2=1/x_0,\ y_2=-((y_0+w_0)x_0+\alpha_2)x_0,\ z_2=z_0-x_0,\ w_2=w_0,\ q_2=q_0,\ p_2=p_0,\\
&r_3:x_3=x_4,\ y_3=y_4,\ z_3=-((z_4-q_4)w_4-\alpha_3)w_4,\ w_3=1/w_4,\ q_3=q_4,\ p_3=p_4+w_4.
\end{aligned}$$

Then such a system coincides with the system (18).

3. THE SYSTEM OF TYPE $B_5^{(1)}$

In this section, we propose two types of a 5-parameter family of coupled Painlevé III systems in dimension six with extended affine Weyl group symmetry of type $B_5^{(1)}$. Each of them is equivalent to a polynomial Hamiltonian system, however, each has a different representaion of type $B_5^{(1)}$. We also show that each of them is equivalent to the system (18) by a birational and symplectic transformation, respectively.

The first member is given by

$$
\left\{
\begin{aligned}
\frac{dx}{dt} &= \frac{2x^2y+2\alpha_0x}{t}-\frac{4xyz-2\alpha_1z}{t},\\
\frac{dy}{dt} &= -\frac{2xy^2+2\alpha_0y-1}{t}+\frac{2y^2z}{t},\\
\frac{dz}{dt} &= \frac{2z^2w+2(\alpha_0+\alpha_1+\alpha_2)z+t}{t}-\frac{2p}{t},\\
\frac{dw}{dt} &= -\frac{2zw^2+2(\alpha_0+\alpha_1+\alpha_2)w+1}{t}+\frac{2(xy-\alpha_1)y}{t},\\
\frac{dq}{dt} &= \frac{2q^2p-tq^2-2(\alpha_0+\alpha_1+\alpha_2+\alpha_3)q}{t}-\frac{2w}{t},\\
\frac{dp}{dt} &= \frac{-2qp^2+2tqp+2(\alpha_0+\alpha_1+\alpha_2+\alpha_3)p+\alpha_4t}{t}
\end{aligned}
\right. \tag{19}
$$

with the Hamiltonian

$$
\begin{aligned}
H &= H_3(x,y,t;\alpha_0)+H_{III}^{D_7^{(1)}}(z,w,t,2(\alpha_0+\alpha_1+\alpha_2))\\
&\quad + H_4(q,p,t;\alpha_4,\alpha_5)-\frac{2(xy-\alpha_1)yz+2wp}{t}\\
&= \frac{x^2y^2+2\alpha_0xy-x}{t}+\frac{z^2w^2+2(\alpha_0+\alpha_1+\alpha_2)zw+z+tw}{t}\\
&\quad + \frac{q^2p^2-tq^2p-2(\alpha_0+\alpha_1+\alpha_2+\alpha_3)qp-\alpha_4tq}{t}-\frac{2(xy-\alpha_1)yz+2wp}{t}.
\end{aligned} \tag{20}
$$

Here x, y, z, w, q and p denote unknown complex variables and $\alpha_0, \alpha_1, \dots, \alpha_5$ are complex parameters satisfying the relation:

$$
2\alpha_0+2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5=1. \tag{21}
$$

THEOREM 3.1. *The system* (19) *admits extended affine Weyl group symmetry of type* $B_5^{(1)}$ *as the group of its Bäcklund transformations* (cf. [2, 8, 9]), *whose generators are explicitly given as follows*:

$$
\begin{aligned}
s_0 &: (*) \rightarrow \left(-x-\frac{2\alpha_0}{y}+\frac{1}{y^2},-y,-z,-w,-q,-p,-t;-\alpha_0,\alpha_1+2\alpha_0,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right),\\
s_1 &: (*) \rightarrow \left(x,y-\frac{\alpha_1}{x},z,w,q,p,t;\alpha_0+\alpha_1,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4,\alpha_5\right),\\
s_2 &: (*) \rightarrow \left(x+\frac{\alpha_2}{y+w},y,z+\frac{\alpha_2}{y+w},w,q,p,t;\alpha_0,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4,\alpha_5\right),\\
s_3 &: (*) \rightarrow \left(x,y,z,w-\frac{\alpha_3q}{zq-1},q,p-\frac{\alpha_3z}{zq-1},t;\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3\right),\\
s_4 &: (*) \rightarrow \left(x,y,z,w,q+\frac{\alpha_4}{p},p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5\right),\\
s_5 &: (*) \rightarrow \left(x,y,z,w,q+\frac{\alpha_5}{p-t},p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_5,\alpha_4,-\alpha_5\right),\\
\varphi &: (*) \rightarrow (x,y,z,w,q,p-t,-t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_4).
\end{aligned}
$$

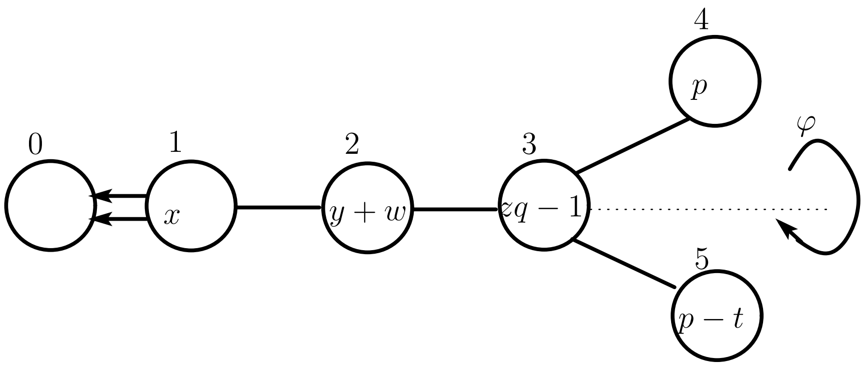

FIGURE 3. Dynkin diagram of type $B_5^{(1)}$

THEOREM 3.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x,y,z,w,q,p]$. *We assume that*

(A1) *deg*$(H)=4$ *with respect to* x,y,z,w,q,p.

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate* r_i $(i=0,1,2,4,5)$:

$$r_0: x_0 = x + \frac{2\alpha_0}{y} - \frac{1}{y^2},\ y_0 = y,\ z_0 = z,\ w_0 = w,\ q_0 = q,\ p_0 = p,$$
$$r_1: x_1 = -(xy-\alpha_1)y,\ y_1 = 1/y,\ z_1 = z,\ w_1 = w,\ q_1 = q,\ p_1 = p,$$
$$r_2: x_2 = 1/x,\ y_2 = -((y+w)x+\alpha_2)x,\ z_2 = z-x,\ w_2 = w,\ q_2 = q,\ p_2 = p,$$
$$r_4: x_4 = x,\ y_4 = y,\ z_4 = z,\ w_4 = w,\ q_4 = 1/q,\ p_4 = -(pq+\alpha_4)q,$$
$$r_5: x_5 = x,\ y_5 = y,\ z_5 = z,\ w_5 = w.\ q_5 = 1/q,\ p_5 = -((p-t)q+\alpha_5)q.$$

(A3) *In addition to the assumption* (A2), *the Hamiltonian system in the coordinate system* $(x_4,y_4,z_4,w_4,q_4,p_4)$ *becomes again a polynomial Hamiltonian system in the coordinate* r_3:

$$r_3: x_3 = x_4,\ y_3 = y_4,\ z_3 = -((z_4-q_4)w_4-\alpha_3)w_4,\ w_3 = 1/w_4,\ q_3 = q_4,\ p_3 = p_4 + w_4.$$

Then such a system coincides with the system (19).

Theorems 3.1 and 3.2 can be checked by a direct calculation, respectively.

THEOREM 3.3. *For the system* (18) *of type* $D_5^{(1)}$, *we make the change of parameters and variables*

$$A_0 = \frac{\alpha_1-\alpha_0}{2},\quad A_1 = \alpha_0,\quad A_2 = \alpha_2,\quad A_3 = \alpha_3,\quad A_4 = \alpha_4,\quad A_5 = \alpha_5, \tag{22}$$

$$X = -(xy-\alpha_0)y,\quad Y = \frac{1}{y},\quad Z = z,\quad W = w,\quad Q = q,\quad P = p \tag{23}$$

from $\alpha_0,\alpha_1,\dots,\alpha_5,x,y,z,w,q,p$ *to* $A_0,A_1,\dots,A_5,X,Y,Z,W,Q,P$. *Then the system* (18) *can also be written in the new variables* X,Y,Z,W,Q,P *and parameters* $A_0,A_1,\dots,A_5$

as a Hamiltonian system. This new system tends to the system (19) *with the Hamiltonian* (20)*.*

PROOF. Notice that

$$2A_0+2A_1+2A_2+2A_3+A_4+A_5=\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5=1$$

and the change of variables from (x,y,z,w,q,p) to (X,Y,Z,W,Q,P) is symplectic. Choose S_i $(i=0,1,\dots,5)$ and φ as

$$S_0:=\pi_1,\ S_1:=s_0,\ S_2:=s_2,\ S_3:=s_3,\ S_4:=s_4,\ S_5:=s_5,\ \varphi:=\pi_2.$$

Then the transformations S_i are reflections of the parameters $A_0,A_1,\dots,A_5$. The transformation group $\tilde{W}(B_5^{(1)})=<S_0,S_1,\dots,S_5,\varphi>$ coincides with the transformations given in Theorem 3.1. □

The second member is given by

$$(24)\quad \begin{cases} \dfrac{dx}{dt}=\dfrac{2x^2y+2xy-(\alpha_0+\alpha_1)x-\alpha_0}{t},\\ \dfrac{dy}{dt}=-\dfrac{2xy^2+y^2-(\alpha_0+\alpha_1)y}{t}-\dfrac{2z}{t},\\ \dfrac{dz}{dt}=\dfrac{2z^2w+(\alpha_0+\alpha_1+2\alpha_2)z+t}{t}+\dfrac{2q(qp+\alpha_4)}{t},\\ \dfrac{dw}{dt}=-\dfrac{2zw^2+(\alpha_0+\alpha_1+2\alpha_2)w+1}{t}-\dfrac{2x}{t},\\ \dfrac{dq}{dt}=\dfrac{2q^2p-(2\alpha_5-1)q+t}{t}+\dfrac{2wq^2}{t},\\ \dfrac{dp}{dt}=\dfrac{-2qp^2+(2\alpha_5-1)p}{t}-\dfrac{4wqp+2\alpha_4w}{t} \end{cases}$$

with the Hamiltonian

$$(25)\quad \begin{aligned} H=&H_1(x,y,t;\alpha_0,\alpha_1)+H_{III}^{D_7^{(1)}}(z,w,t,\alpha_0+\alpha_1+2\alpha_2)\\ &+H_2(q,p,t;\alpha_5)+\frac{2xz+2wq(qp+\alpha_4)}{t}\\ =&\frac{x^2y^2+xy^2-(\alpha_0+\alpha_1)xy-\alpha_0y}{t}+\frac{z^2w^2+(\alpha_0+\alpha_1+2\alpha_2)zw+z+tw}{t}\\ &+\frac{q^2p^2-(2\alpha_5-1)qp+tp}{t}+\frac{2xz+2wq(qp+\alpha_4)}{t}. \end{aligned}$$

Here x,y,z,w,q and p denote unknown complex variables and $\alpha_0,\alpha_1,\dots,\alpha_5$ are complex parameters satisfying the relation:

$$(26)\quad \alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5=1.$$

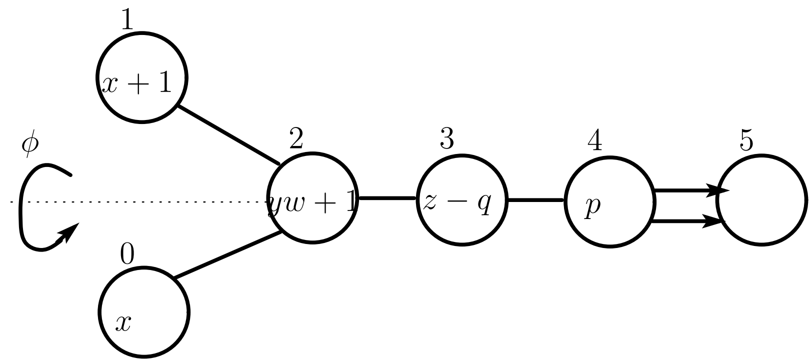

FIGURE 4. Dynkin diagram of type $B_5^{(1)}$

THEOREM 3.4. *The system* (24) *admits extended affine Weyl group symmetry of type* $B_5^{(1)}$ *as the group of its Bäcklund transformations* (cf. [2, 8, 9]), *whose generators are explicitly given as follows*:

$$\begin{aligned}
&s_0 : (*) \to (x, y - \frac{\alpha_0}{x}, z, w, q, p, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5),\\
&s_1 : (*) \to (x, y - \frac{\alpha_1}{x+1}, z, w, q, p, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5),\\
&s_2 : (*) \to (x + \frac{\alpha_2 w}{yw+1}, y, z + \frac{\alpha_2 y}{yw+1}, w, q, p, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5),\\
&s_3 : (*) \to (x, y, z, w - \frac{\alpha_3}{z-q}, q, p + \frac{\alpha_3}{z-q}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5),\\
&s_4 : (*) \to (x, y, z, w, q + \frac{\alpha_4}{p}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4),\\
&s_5 : (*) \to (x, y, z, w, q, p - \frac{2\alpha_5}{q} + \frac{t}{q^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + 2\alpha_5, -\alpha_5),\\
&\phi : (*) \to (-x-1, -y, -z, -w, -q, -p, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5).
\end{aligned}$$

THEOREM 3.5. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w, q, p]$. *We assume that*

(A1) $deg(H) = 4$ *with respect to* x, y, z, w, q, p.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, 3, 4, 5)$:

$$\begin{aligned}
&r_0 : x_0 = -(xy - \alpha_0)y,\ y_0 = 1/y,\ z_0 = z,\ w_0 = w,\ q_0 = q,\ p_0 = p,\\
&r_1 : x_1 = -((x+1)y - \alpha_1)y,\ y_1 = 1/y,\ z_1 = z,\ w_1 = w,\ q_1 = q,\ p_1 = p,\\
&r_3 : x_3 = x,\ y_3 = y,\ z_3 = -((z-q)w - \alpha_3)w,\ w_3 = 1/w,\ q_3 = q,\ p_3 = p + w,\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = z,\ w_4 = w,\ q_4 = 1/q,\ p_4 = -(pq + \alpha_4)q,\\
&r_5 : x_5 = x,\ y_5 = y,\ z_5 = z,\ w_5 = w.\ q_5 = q,\ p_5 = p - \frac{2\alpha_5}{q} + \frac{t}{q^2}.
\end{aligned}$$

(A3) *In addition to the assumption* (A2), *the Hamiltonian system in the coordinate system* $(x_0, y_0, z_0, w_0, q_0, p_0)$ *becomes again a polynomial Hamiltonian system in the coordinate* r_2:

$$r_2 : x_2 = 1/x_0,\ y_2 = -((y_0 + w_0)x_0 + \alpha_2)x_0,\ z_2 = z_0 - x_0,\ w_2 = w_0,\ q_2 = q_0,\ p_2 = p_0.$$

Then such a system coincides with the system (24).

Theorems 3.4 and 3.5 can be checked by a direct calculation, respectively.

THEOREM 3.6. *For the system* (18) *of type* $D_5^{(1)}$, *we make the change of parameters and variables*

$$A_0 = \alpha_0, \quad A_1 = \alpha_1, \quad A_2 = \alpha_2, \quad A_3 = \alpha_3, \quad A_4 = \alpha_4, \quad A_5 = \frac{\alpha_5 - \alpha_4}{2}, \tag{27}$$

$$X = x, \quad Y = y, \quad Z = z, \quad W = w, \quad Q = \frac{1}{q}, \quad P = -(pq + \alpha_4)q \tag{28}$$

from $\alpha_0, \alpha_1, \dots, \alpha_5, x, y, z, w, q, p$ *to* $A_0, A_1, \dots, A_5, X, Y, Z, W, Q, P$. *Then the system* (18) *can also be written in the new variables* X, Y, Z, W, Q, P *and parameters* $A_0, A_1, \dots, A_5$ *as a Hamiltonian system. This new system tends to the system* (24) *with the Hamiltonian* (25).

PROOF. Notice that

$$A_0 + A_1 + 2A_2 + 2A_3 + 2A_4 + 2A_5 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1$$

and the change of variables from (x, y, z, w, q, p) to (X, Y, Z, W, Q, P) is symplectic. Choose S_i $(i = 0, 1, \dots, 5)$ and φ as

$$S_0 := s_0,\ S_1 := s_1,\ S_2 := s_2,\ S_3 := s_3,\ S_4 := s_4,\ S_5 := \pi_2,\ \phi := \pi_1.$$

Then the transformations S_i are reflections of the parameters $A_0, A_1, \dots, A_5$. The transformation group $\tilde{W}(B_5^{(1)}) =< S_0, S_1, \dots, S_5, \phi >$ coincides with the transformations given in Theorem 3.4. □

By using Theorems 3.3 and 3.6, we show the relation between the system (19) and the system (24).

THEOREM 3.7. *For the system* (19) *of type* $B_5^{(1)}$, *we make the change of parameters and variables*

$$A_0 = \alpha_1, \quad A_1 = 2\alpha_0 + \alpha_1, \quad A_2 = \alpha_2, \quad A_3 = \alpha_3, \quad A_4 = \alpha_4, \quad A_5 = \frac{\alpha_5 - \alpha_4}{2}, \tag{29}$$

$$X = -(xy - \alpha_1)y, \quad Y = \frac{1}{y}, \quad Z = z, \quad W = w, \quad Q = \frac{1}{q}, \quad P = -(pq + \alpha_4)q \tag{30}$$

from $\alpha_0, \alpha_1, \dots, \alpha_5, x, y, z, w, q, p$ *to* $A_0, A_1, \dots, A_5, X, Y, Z, W, Q, P$. *Then the system* (19) *can also be written in the new variables* X, Y, Z, W, Q, P *and parameters* $A_0, A_1, \dots, A_5$ *as a Hamiltonian system. This new system tends to the system* (24) *with the Hamiltonian* (25).

4. The system of type $D_6^{(2)}$

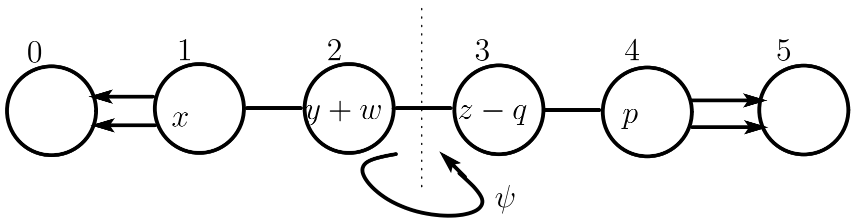

Figure 5. Dynkin diagram of type $D_6^{(2)}$

In this section, we propose a 5-parameter family of coupled Painlevé III systems in dimension six with extended affine Weyl group symmetry of type $D_6^{(2)}$ given by

$$
(31)\quad \begin{cases}
\dfrac{dx}{dt} = \dfrac{2x^2y+2\alpha_0x}{t} - \dfrac{2(2xyz-\alpha_1z)}{t},\\
\dfrac{dy}{dt} = -\dfrac{2xy^2+2\alpha_0y-1}{t} + \dfrac{2y^2z}{t},\\
\dfrac{dz}{dt} = \dfrac{2z^2w+2(\alpha_0+\alpha_1+\alpha_2)z+t}{t} + \dfrac{2q(qp+\alpha_4)}{t},\\
\dfrac{dw}{dt} = -\dfrac{2zw^2+2(\alpha_0+\alpha_1+\alpha_2)w+1}{t} + \dfrac{2y(xy-\alpha_1)}{t},\\
\dfrac{dq}{dt} = \dfrac{2q^2p-(2\alpha_5-1)q+t}{t} + \dfrac{2wq^2}{t},\\
\dfrac{dp}{dt} = \dfrac{-2qp^2+(2\alpha_5-1)p}{t} - \dfrac{2(2wqp+\alpha_4w)}{t}
\end{cases}
$$

with the Hamiltonian

$$
(32)\quad \begin{aligned}
H &= H_3(x,y,t;\alpha_0) + H_{III}^{D_7^{(1)}}(z,w,t,2(\alpha_0+\alpha_1+\alpha_2))\\
&\quad + H_2(q,p,t;\alpha_5) + \frac{2(wq(qp+\alpha_4)-yz(xy-\alpha_1))}{t}\\
&= \frac{x^2y^2+2\alpha_0xy-x}{t} + \frac{z^2w^2+2(\alpha_0+\alpha_1+\alpha_2)zw+z+tw}{t}\\
&\quad + \frac{q^2p^2-(2\alpha_5-1)qp+tp}{t} + \frac{2(wq(qp+\alpha_4)-yz(xy-\alpha_1))}{t}.
\end{aligned}
$$

Here x, y, z, w, q and p denote unknown complex variables and $\alpha_0, \alpha_1, \dots, \alpha_5$ are complex parameters satisfying the relation:

$$
(33)\quad \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 = \frac{1}{2}.
$$

Theorem 4.1. *The system* (31) *admits extended affine Weyl group symmetry of type* $D_6^{(2)}$ *as the group of its Bäcklund transformations* (cf. [2, 8, 9]), *whose generators are*

explicitly given as follows:

$$
\begin{aligned}
&s_0 : (*) \rightarrow (-x - \frac{2\alpha_0}{y} + \frac{1}{y^2}, -y, -z, -w, -q, -p, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5),\\
&s_1 : (*) \rightarrow (x, y - \frac{\alpha_1}{x}, z, w, q, p, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5),\\
&s_2 : (*) \rightarrow (x + \frac{\alpha_2}{y+w}, y, z + \frac{\alpha_2}{y+w}, w, q, p, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5),\\
&s_3 : (*) \rightarrow (x, y, z, w - \frac{\alpha_3}{z-q}, q, p + \frac{\alpha_3}{z-q}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5),\\
&s_4 : (*) \rightarrow (x, y, z, w, q + \frac{\alpha_4}{p}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4),\\
&s_5 : (*) \rightarrow (x, y, z, w, q, p - \frac{2\alpha_5}{q} + \frac{t}{q^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + 2\alpha_5, -\alpha_5),\\
&\psi : (*) \rightarrow (-tp, \frac{q}{t}, tw, -\frac{z}{t}, -ty, \frac{x}{t}, -t; \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0).
\end{aligned}
$$

THEOREM 4.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w, q, p]$. We assume that*

(A1) $deg(H) = 4$ with respect to x, y, z, w, q, p.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, \dots, 5)$:

$$
\begin{aligned}
&r_0 : x_0 = x + \frac{2\alpha_0}{y} - \frac{1}{y^2},\ y_0 = y,\ z_0 = z,\ w_0 = w,\ q_0 = q,\ p_0 = p,\\
&r_1 : x_1 = -(xy - \alpha_1)y,\ y_1 = 1/y,\ z_1 = z,\ w_1 = w,\ q_1 = q,\ p_1 = p,\\
&r_2 : x_2 = 1/x,\ y_2 = -((y + w)x + \alpha_2)x,\ z_2 = z - x,\ w_2 = w,\ q_2 = q,\ p_2 = p,\\
&r_3 : x_3 = x,\ y_3 = y,\ z_3 = -((z - q)w - \alpha_3)w,\ w_3 = 1/w,\ q_3 = q,\ p_3 = p + w,\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = z,\ w_4 = w,\ q_4 = 1/q,\ p_4 = -(pq + \alpha_4)q,\\
&r_5 : x_5 = x,\ y_5 = y,\ z_5 = z,\ w_5 = w.\ q_5 = q,\ p_5 = p - \frac{2\alpha_5}{q} + \frac{t}{q^2}.
\end{aligned}
$$

Then such a system coincides with the system (31).

Theorems 4.1 and 4.2 can be checked by a direct calculation, respectively.

THEOREM 4.3. *For the system* (18) *of type $D_5^{(1)}$, we make the change of parameters and variables*

$$
A_0 = \frac{\alpha_1 - \alpha_0}{2}, \quad A_1 = \alpha_0, \quad A_2 = \alpha_2, \quad A_3 = \alpha_3, \quad A_4 = \alpha_4, \quad A_5 = \frac{\alpha_5 - \alpha_4}{2}, \tag{34}
$$

$$
X = -(xy - \alpha_0)y, \quad Y = \frac{1}{y}, \quad Z = z, \quad W = w, \quad Q = \frac{1}{q}, \quad P = -(pq + \alpha_4)q \tag{35}
$$

from $\alpha_0, \alpha_1, \dots, \alpha_5, x, y, z, w, q, p$ to $A_0, A_1, \dots, A_5, X, Y, Z, W, Q, P$. Then the system (18) *can also be written in the new variables X, Y, Z, W, Q, P and parameters $A_0, A_1, \dots, A_5$*

as a Hamiltonian system. This new system tends to the system (31) *with the Hamiltonian* (32)*.*

Proof. Notice that

$$2(A_0 + A_1 + A_2 + A_3 + A_4 + A_5) = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1$$

and the change of variables from (x, y, z, w, q, p) to (X, Y, Z, W, Q, P) is symplectic. Choose S_i $(i = 0, 1, \dots, 5)$ and φ as

$$S_0 := \pi_1,\ S_1 := s_0,\ S_2 := s_2,\ S_3 := s_3,\ S_4 := s_4,\ S_5 := \pi_2,\ \psi := \pi_1\pi_2\pi_3.$$

Then the transformations S_i are reflections of the parameters $A_0, A_1, \dots, A_5$. The transformation group $\tilde{W}(D_6^{(2)}) =< S_0, S_1, \dots, S_5, \psi >$ coincides with the transformations given in Theorem 4.1. □

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